Philosophia Mathematica (III) **18** (2010), 227–252.

Critical Studies/Book Reviews

JEAN-PIERRE MARQUIS. ***From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory***. Logic, Epistemology, and the Unity of Science; 14. Springer, 2009. ISBN 978-1-4020-9383-8 (hbk). Pp. x + 310.

Reviewed by MOLLY KAO, NICOLAS FILLION, AND JOHN BELL*

Jean-Pierre Marquis has written a number of interesting and original papers on the philosophical issues related to category theory (*e.g.*, [Marquis, 1995; Marquis, 1998; Landry and Marquis, 2005; Marquis, 2006]). His recent book incorporates a number of themes that he has previously examined, but considered from a more definite perspective: in this book his principal objective is to establish the claim that category theory is a generalization of Felix Klein's Erlangen program. The central tenet he urges here is that category theory should be thought of in essentially *geometric* terms. The philosophically precise and innovative way in which he develops this thesis makes the book relevant to all those with some interest in category theory, logic, the foundations of mathematics, and more particularly the interplay among them. Readers should include all members of Marquis' intended audience—mathematicians, philosophers and historians alike. Although Marquis' writing is clear and accessible, this book is primarily for those with some familiarity with the subject matter. Readers lacking sufficient background in logic, geometry, algebra, and category theory should expect to do some supplemental reading in order to assimilate the arguments; but it would surely be a worthwhile endeavour for anyone interested in the history and philosophy of category theory and its relation to the more general body of mathematics. This review will provide a summary of the main philosophical points developed by Marquis.

1. The Foundations of Mathematics

Category theory was introduced in 1945 by Eilenberg and Mac Lane as a language for describing and organizing mathematical constructions, for example the construction of homology groups. Later, largely through the efforts of F.W. Lawvere, a new and major stimulus for the development of category theory emerged, namely: can it provide a foundation for mathematics, and how exactly should this be achieved? As is always the case with foundations of mathematics, this problem has many intricate and

* Department of Philosophy, The University of Western Ontario, London, Ontario N6A 3K7, Canada. mkao4@uwo.ca, nicolas.fillion@gmail.com, jbell@uwo.ca

far-reaching ramifications. Marquis judiciously interlocks or disentangles them, according to the case. In a former paper [1995], he delineates several different ways in which a system might be considered foundational for mathematics: logical, cognitive, epistemological, semantical, ontological, and methodological. As he explains, a philosophy of mathematics is an ordering of the above foundational relations. Within the confines of a particular philosophy of mathematics, certain of these relations lose their foundational status if it is maintained that they are reducible to others or are of no consequence. This point has great import for the debate on the foundational status of category theory, as he explains:

> [. . .] some of the arguments given either in favor or against category theory are based on different conceptions of what should be included in, or what should be meant by, the foundations of mathematics. It is hoped that this will allow us to see precisely where the different parties disagree and, from there, orient the debate appropriately. [Marquis, 1995, pp. 421–422]

One of the most important contributions of this book is, undoubtedly, the clarity with which it isolates the various aspects of foundations of mathematics. In his effort to explain the foundational significance of Klein's program and the way in which category theory extends it, Marquis deploys a number of concepts and distinctions that reveal important complementarities among logical, epistemological, semantical, ontological, and methodological questions. Moreover, the rich context provided by the algebraic perspective inherent in Klein's program enables Marquis to offer important insights into the foundational role of logic, semantics, and ontology. This aspect of Marquis' work is important not just for philosophical issues concerning category theory, but for the philosophy of mathematics in general.

2. Klein's Erlangen Program

2.1. *Towards Transformation Groups*

Marquis' characterization of Klein's program focuses on the three aspects most relevant to his claim that category theory is a generalization of that program, namely, the way in which transformation groups encode criteria of identity for geometric objects, how they provide criteria of identity for geometric properties, and the importance, as a consequence, of Klein's classification of geometric spaces for foundations.

One of Klein's principal goals was to unify the study of different geometric spaces. At the time, the connections among different geometries were far from clear. In fact, some seemed to be in conflict, *e.g.*, projective geometry seemed to be incompatible with the metric approaches of

Euclidean geometry. In order to obtain such a classification, Klein proposed a framework characterizing the intrinsic properties of geometric spaces, thereby enabling them to be describable in a common manner. He accomplished this by showing that transformation groups represent the essential properties of a geometry. For example, the group of isometries on the Euclidean plane can be used to express the notion of congruence, a basic relation in Euclidean geometry. Since the transformation group is an *algebraic* object, this meant that one could characterize a geometric space through its algebraic structure, independently of its constitution as a space.

Marquis gives an elegant and instructive explanation of how transformation groups figure centrally in Klein's program. For instance, he illustrates what it means for a geometry to be a subgeometry in the algebraic sense by showing how the group of Euclidean transformations forms a subgroup of the group of affine transformations, so that Euclidean geometry is a subgeometry of affine geometry. In this way, it is made clear that Klein's breakthrough was the recognition that one can use the algebraic concept of group to encode the basic relations among geometries, since the formal properties of geometries are shown to be contained in the algebraic properties of their transformation groups.

Klein's unifying framework for the study of elementary geometry is accordingly algebraic; it amounts to the identification of a skeleton for the body of a geometric space. Nevertheless, despite this talk of logical form and skeleton of geometries, Marquis suggests that it is 'a perfectly reasonable claim' that 'transformation groups cannot be taken as being *logically* fundamental in geometry' (p. 39).[1] Still, they can be considered fundamental in a different sense, since the idea of 'form' can be captured in a number of different ways. While the traditional view is that logical form provides all the crucial information about a system, Marquis claims that other conceptions of a foundation can be equally meaningful. While logical foundations provide a set list of axioms and are therefore presumed to be epistemically foundational—insofar as they provide the building blocks on which theories are built—this feature is not demanded of an algebraic approach. He argues that an algebraic foundation serving to reveal the stucture of a space can also be epistemically efficacious, for it generates both an explanation and better understanding (p. 40). An algebraic foundation can foster new developments in mathematics, *via* the relation of fundamental concepts between different branches of mathematics, which indicates that it is fruitful as a methodological foundation.[2]

[1] Unattributed page numbers are from the book under review.

[2] Marquis is here expanding on a theme that he previously developed in [1998].

2.2. *Criteria of Identity*

A key element of Klein's program was to furnish a criterion of identity for geometric *objects*. For instance, two geometrical figures can be considered identical when one can be obtained from the other by applying a series of transformations. Marquis fleshes out Klein's point by means of the type-token distinction. Geometry is concerned, he says, not with tokens, but with types, and what we are characterizing when we postulate a criterion of identity are types of geometric figures. That is, we are interested in geometrical figures insofar as they possess certain general properties, not in particular circles or triangles.[3] The transformation group provides us with a criterion for determining when two objects are identical. Different transformation groups give rise to different criteria of identity. This constitutes an *ontological foundation* in the following sense:

> [W]e need to know in some sense what the theory is about or what it is to be an object referred to by the expressions of the theory. This is a basic ontological requirement and transformation groups are intimately related to this requirement. (p. 19)

Transformation groups not only reveal the inherent structure of a geometry, but also yield information about the elements of the group. This conception of an ontological foundation is decidedly different from the traditional one.

Moreover, Klein pointed out that the transformation group can be used to identify geometric *properties* in a space: a property P of a figure should be considered geometric if and only if it can be considered independently of any particular co-ordinate system. That is, whenever a figure possesses P and a transformation from the group is applied, if the transformed figure still possesses P, then P is a *geometric* property of that space, *i.e.*, P is a geometric property exactly when it is *invariant* under characteristic transformations (p. 22).

Since transformations act on the whole space rather than on particular objects within it, the identity of geometric spaces can also be encoded in this way. Thus, spaces are *geometrically identical* if there is a structure-preserving bijection—an isomorphism—between them. Two geometric spaces may look very different—*e.g.*, one may deal with spheres and another with lines—while still being fundamentally structurally identical. Moreover, by transference, the theorems provable within one geometry will be provable in the other. As a result, transformation groups can characterize

[3] In this context, a particular figure (token) is merely a set of points. Thus, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ and $\{(x, y) \in \mathbb{R}^2 : (x - h)^2 + (y - k)^2 = r^2\}$ are different token-circles, whereas they are algebraically identical under the group of isometries, *i.e.*, they are of the same type.

objects, properties of objects, geometries, and properties of geometries. As such, they provide an ontological foundation for geometry.

3. Category Theory as an Extension of Klein's Program

We now come to what may be considered the central argument of Marquis' book, namely, that category theory is a generalization of Klein's program. Interestingly, Eilenberg and Mac Lane themselves considered category theory to be a generalization of Klein's program. However, Marquis advances the discussion by showing that, although their argument was erroneous, category theory is nevertheless such a generalization, albeit different from the one initially intended.

According to Marquis, Eilenberg's and Mac Lane's mistake was their identification of the relation between a transformation group and its subgroups as the crucial aspect of Klein's program. While this relation is an important concept in both contexts, Marquis claims that Eilenberg and Mac Lane were not approaching the analogy in the right way. He points out that their paper discussed only the categories of groups and Abelian groups, whose properties are not easily if at all generalizable to categories in general. However, according to Marquis, it was the rigorous definition of a category that constituted the main breakthrough enabling category theory to be recognized as a generalization of Klein's program. He maintains, with good reason, that the relationship between abstract categories and mathematical structures is analogous to the relationship between the concept of transformation group and geometric structures (p. 102).

Marquis draws and scrutinizes several analogies between category theory and Klein's program; in particular, the analogy between inverse operations in geometry and in category theory, and that between additive categories and Lie groups. While transformations in geometry refer to a mathematical displacement of geometric objects, (mutually) inverse operations in category theory embody the idea that functors represent *conceptual* transformations, in the sense that they are invariant mathematical constructions (p. 152). The most important among those inverse operations are adjoint functors. Marquis suggests that the birth of pure category theory took place with Kan's paper which marks the first explicit appearance of the notion of adjoint functor:

> It is my belief that the claim that category theory is a generalization of Klein's program takes a new meaning when adjoint functors are introduced and when their significance is understood. (p. 138)

Marquis devotes chapter 5 to explaining how categories can be regarded as spaces, and functors as types of transformations. That functors allow us to talk about conceptual transformations and invariant content is a crucial aspect of his main thesis. It was Kan who first transcended the idea that category theory was no more than a useful language for obtaining results in other branches of mathematics. He saw that there were interesting results to be had from studying categories, functors, and relations among them in their own right. Marquis provides an extensive historical background of Kan's discovery of adjoint functors in homotopy theory, and how his line of research furthers Eilenberg's and Mac Lane's work. Kan took many different notions from homotopy theory and defined them in terms of functors on categories, thus showing what content certain categories have in common. Even more striking was Kan's unification of these results into a simpler and more elegant presentation, through which he revealed hitherto unsuspected connections between different mathematical concepts. For Marquis, it is this which provides the central analogy with Klein's program.

4. Category Theory and the Foundations of Mathematics

It was F.W. Lawvere who conceived the bold idea that category theory could provide a foundation for mathematics, in the sense that all of mathematics could be developed within the category of categories. However, as he explained, this category should be considered a foundation in an *open* manner because he believed that 'there is no ultimate and once-and-for-all foundational framework' (p. 191). He also suggested that this approach to foundational issues in mathematics was closely linked to mathematical practice, since mathematical constructions can be considered as functors between different categories. In chapter 6, Marquis explains how, through this program, the guiding principle 'look for adjoints to given functors' became central for both foundational and practical endeavours.

Lawvere's axiomatization of the category of categories continues to animate discussions concerning category theory in the foundations of mathematics. Topos theory—which Marquis discusses in the context of geometric logic—plays a central role in that argument. The most striking thing about topos theory is the power it derives from its generality and applicability to various branches of mathematics. Grothendieck has shown that toposes can be identified with spaces. In chapter 7, Marquis explains how this provides further grounds for his thesis that category theory is an extension of Klein's program. His presentation of the foundational programs based on category theory and topos theory is, as throughout the book, stimulating and insightful.

Marquis puts forward certain objections to these foundational programs, to which he provides original answers. In its standard definition and axiomatization, category theory is described in terms of objects and morphisms between them. This has led to the idea that category theory is irreducibly dependent on the idea of a collection, which in turn would seem to presuppose the existence of sets and the soundness of set theory. Typically, arguments against the foundational status of category theory claim that it cannot be developed without presupposing set theory. Marquis offers several rebuttals to this argument. One is to point out that while categories can be considered structures, they are not merely one type of structure among others: category theory is a general theory of structures, and so is, in essence, a metastructure (p. 55). In addition, he invokes the difference between what he identifies as cognitive explanatory priority as opposed to logical priority. Contrary to the assumption inherent in the above objection, he remarks that while it may be an empirical fact that collections are cognitively prior to categories, this is merely a feature of our psychological makeup. It does not imply that sets or collections are logically prior to the concept of a category. Indeed, since categories or structures are normally presented abstractly, they do not need to be underpinned by the concept of a set or collection. As Marquis points out, a parallel argument could be made concerning algebraic transformation groups being foundational for geometric spaces, but he urges that this conflates the question of logical priority with epistemological priority and cognitive value. In any event, the idea of a collection is much more general than the technical definition of a set, so that it is by no means the case that the concept of collection presupposes the concept of set. Indeed, if cognitive considerations are put aside, the argument seems to go the other way. For Lawvere has shown that set theory can be presented categorically, and this kind of organization allows category theory to do, at the mathematical level, the important work done by set theory, at the same time eliminating its extraneous elements (p. 209). The *logical* foundational status of category theory must then be set at least as high as that of set theory. Here Marquis' insightful distinction among the various sorts of foundations comes into its own.

Marquis also engages Kreisel's argument against the foundational status of category theory. It was in fact Kreisel who first offered an analysis of the notion of 'foundations' for mathematics that led to a distinction between the justificatory and the merely 'organizational'. In particular, a mathematical foundation should be justificatory, in the sense that it explains or justifies the choice of axioms. Marquis points out that it is difficult to see how some traditionally accepted 'foundations' for mathematics play this role. It is difficult for example to imagine what it would mean for the system of Dedekind cuts to provide reasons for the choice of axioms for the real numbers. More importantly, against Kreisel, Marquis argues that a 'mere' organization of concepts is actually an important task, and if one were to accept a methodological account of foundations, it would be one of the main tasks of that foundation. The discussion of foundations leads to what is perhaps the most important lesson to be drawn from Marquis' book: programs attributing foundational importance to category theory are based on a larger conception of what should constitute foundations for mathematics.

5. Conclusion

Marquis' book is a persuasive synthesis of the history and philosophy of a complex mathematical theory. This is nicely demonstrated by his discussion of arguably the most important concept of category theory—that of adjunctions: he spends a roughly equal amount of space describing the historical development of the crucial discovery of adjoint functors, the technical definition of adjointness, and the philosophical import of adjointness in foundations. Technical and mathematical points are always presented with a particular philosophical point in mind. The book covers a broad range of literature, including many if not all of the seminal papers on category theory. Marquis' writing is clear and compelling.

In his preface, Marquis prejudges the reaction to his book as follows:

> Historians will probably find my historical contributions obvious, simple-minded and narrow-minded ... Mathematicians will probably find many mathematical mistakes and misunderstandings in my presentation and discussion of mathematical concepts and theorems, simplifications of important ideas and results and a lack of a truly global mathematical perspective ... [P]hilosophers of mathematics will assuredly find my philosophical contribution shallow and irrelevant. (p. 8)

This seems unduly self-effacing. On the contrary, the author has succeeded in synthesizing all three of these aspects with clarity and conviction. His book not only situates the work of category theorists in a broader mathematical context; it is likely to open paths for future investigation in both mathematics and philosophy.

References

Landry, E., and J.-P. Marquis [2005]: 'Categories in context: Historical, foundational, and philosophical', *Philosophia Mathematica* (3) **13**, 1–43.

Marquis, J.-P. [1995]: 'Category theory and the foundations of mathematics: Philosophical excavations', *Synthese* **103**, 421–447.

——— [1998]: 'Epistemological aspects of the application of mathematics to itself', in: D. Anapolitanos, A. Baltas, and S. Tsinorema, eds, *Philosophy and the Many Faces of Science*, pp. 183–195. Lanham, Maryland: Rowman and Littlefield.

——— [2006]: 'Categories, sets and the nature of mathematical entities', in J. van Bentham, G. Heinzmann, M. Rebuschi, and H. Visser, eds, *The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today*, pp. 181–192. Berlin: Springer.

doi:10.1093/philmat/nkq006
Advance Access publication March 10, 2010